Section 5.1 / 5.3 / 5.4

5.4 pg 420 #16 $g(x) = \dfrac{x^3}{x^2-3}$

Step 1 Find 1st derivative, so use Quotient Rule b/c division

$f = x^3 \qquad g = x^2 - 3$

$f' = 3x^2 \qquad g' = 2x$

Formula: $\dfrac{f'g - g'f}{g^2} = \dfrac{3x^2(x^2-3) - 2x(x^3)}{(x^2-3)^2}$;

$= \dfrac{3x^4 - 9x^2 - 2x^4}{(x^2-3)^2} = \boxed{\dfrac{x^4 - 9x^2}{(x^2-3)^2}}$ or $= \dfrac{x^2(x^2-9)}{(x^2-3)^2}$

$G'(x) =$

Easiest format to find 2nd derivative

Step 2 Find 2nd derivative

$f = x^4 - 9x^2 \qquad g = (x^2-3)^2$

$f' = 4x^3 - 18x \qquad g' = 2(x^2-3)\cdot 2x = 4x(x^2-3)$

Formula: $\dfrac{f'g - g'f}{g^2} = \dfrac{(4x^3-18x)(x^2-3)^{\cancel{2}} - 4x\cancel{(x^2-3)}(x^4-9x^2)}{((x^2-3)^2)^2}$

$= \dfrac{(x^2-3)\left[(4x^3-18x)(x^2-3) - 4x(x^4-9x^2)\right]}{(x^2-3)^4}$

$= \left[4x^5 - 12x^3 - 18x^3 + 54x - 4x^5 + 36x^3\right]$ FOILED

Then simplify!!

$$= \frac{\cancel{4x^5} - 12x^3 - 18x^3 + 54x - \cancel{4x^5} + 36x^3}{(x^2-3)^{3}}$$

B/c outer term was (x^2-3) took away 1

Now cancel any from above !!

$$\boxed{g''(x) = \frac{6x^3+54x}{(x^2-3)^3} ; \quad = \frac{6x(x^2+9)}{(x^2-3)^3}}$$

So

$$g(x) = \frac{x^3}{x^2-3} \qquad y_1$$

$$g'(x) = \frac{x^4-9x^2}{(x^2-3)^2} = \frac{x^2(x+3)(x-3)}{(x^2-3)^2} \qquad y_2$$

$$g''(x) = \frac{6x^3+54x}{(x^2-3)^3} = \frac{6x(x^2+9)}{(x^2-3)^3} \qquad y_3$$

Step 3 | 1st Derivative Analysis

$g'(x) = \varnothing$ (stationary)

* B/c of denominator we can ignore denominator while finding numerator $= \varnothing$

so $x^2(x+3)(x-3) = \varnothing$

$x^2 = 0 \quad x+3=0 \quad x-3=\varnothing$

$x=0 \quad x=-3 \quad x=3$

3 stationary points (CP)

$g'(x)$ = undefined (singular)

* Care about setting denominator $= \varnothing$, so

$x^2 - 3 = \varnothing$

$x^2 = 3$

$x = \pm\sqrt{3}$

1 singular point

y_2 G' # line

	-3		$-\sqrt{3}$		0		$\sqrt{3}$		3	
+	max	−		−		−		−	min	+
	Stationary		$=-1.73$				$=1.73$		(Stationary)	

always choose max/min from CP not test points

Test Points	#	Sign
$g'(-4) =$	.66	+
$g'(-2) =$	−20	−
$g'(-1) =$	−2	−
$g'(1) =$	−2	−
$g'(2) =$	−20	−
$g'(4) =$	.66	+

Summary

CP ⟹ $-3, -\sqrt{3}, \varnothing, \sqrt{3}, 3$

max ⟹ -3 stationary ∩ ← b/c (+) to (−)

min ⟹ 3 stationary ∪ ← b/c (−) to (+)

Step 4 2nd Derivative analysis

* Know that the points @ $-\sqrt{3}$, $\varnothing$, $\sqrt{3}$ will be included in 2nd derivative analysis, so we can confirm if they are in fact IP (inflection points).

$$g'' = \frac{6x(x^2+9)}{(x^2-3)^3}$$

$g'' = \emptyset$

num $= \emptyset$

$6x(x^2+9) = \emptyset$

$6x = \emptyset$

$x = \emptyset$

$\cancel{x^2+9=\emptyset}$

$\cancel{-9 \quad -9}$

$\cancel{x^2 = -9}$

$\cancel{x = \pm\sqrt{-9}}$

$\cancel{x = \pm 3i}$

NOTE: imaginary #'s do not appear on a real graph

g'' = undefined

den $= \emptyset$

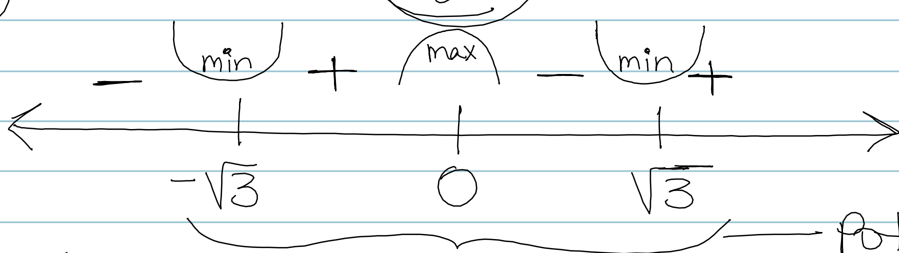
$(x^2-3)^3 = \emptyset$

$x^2 - 3 = \emptyset$

$x^2 = 3$

$x = \pm\sqrt{3}$

g'' # line $\quad y_3$



Potential inflection points

B/C we have 3 CP, we need 4 Test points (3+1)

$g''(-2) = -156 \quad (-)$

$g''(-1) = 7.5 \quad (+)$

$g''(1) = -7.5 \quad (-)$

$g''(2) = 156 \quad (+)$

Summary:

Potential IP $\Rightarrow -\sqrt{3}, \emptyset, \sqrt{3}$

NOTE: *do not need to know or note about stationary/singular when dealing w/ 2nd derivative

$\boxed{\text{STEP 5}}$ B/C we have a denominator in $g(x)$ we have additional things to check.

1st: Check vertical asymptotes (VA)

Where denominator of original $g(x) = \emptyset$

, so $g(x) = \frac{x^3}{x^2-3}$, take denominator

$x^2 - 3 = \emptyset$

$x^2 = 3$

$x = \pm\sqrt{3}$

VA

2nd: Check horizontal asymptotes (HA)

what we are actually checking

$\lim\limits_{x \to \infty}$

Ex: $f(x) = \frac{ax^n + \ldots}{bx^d + \ldots} =$

, so if $n > d$ = no HA, but other lines may exist

Remember Rules

$n = d$ so $y = \frac{a}{b}$

$n < d$ $y = \emptyset$

, so $\frac{x^3}{x^2 - 3}$; $\frac{x^3}{x^2} = x$, b/c $x^3 > x^2$ we know there is no HA.

Line looks like this

Use long division to ~~solve~~ solve for oblique asymptote:

$$\begin{array}{r} \cancel{x+3} \quad x+3 \\ x^2-3 \,\big)\, x^3+0x^2+0x+0 \\ \ominus\; x^3 \overset{\oplus}{-} 3x^2 \\ \hline 3x^2+0x+0 \\ \ominus\; 3x^2 \overset{\oplus}{-} 9 \\ \hline 9 \end{array}$$

Ex:
$$\begin{array}{r} 34 \\ 25 \,\big)\, 874 \\ \ominus 75 \downarrow \\ \hline 124 \\ \ominus 100 \\ \hline 24 \end{array}$$

Redo:
$$\begin{array}{r} x \\ x^2-3 \,\big)\, x^3+0x^2+0x+0 \\ \ominus\; x^3 \qquad \overset{\oplus}{-}3x \\ \hline 3x+0 \end{array}$$

x w/remainder of $\frac{3x+0}{x^2-3} = \frac{3x}{x^2-3}$

$y = x$ equation for asymptote — our oblique asymptote

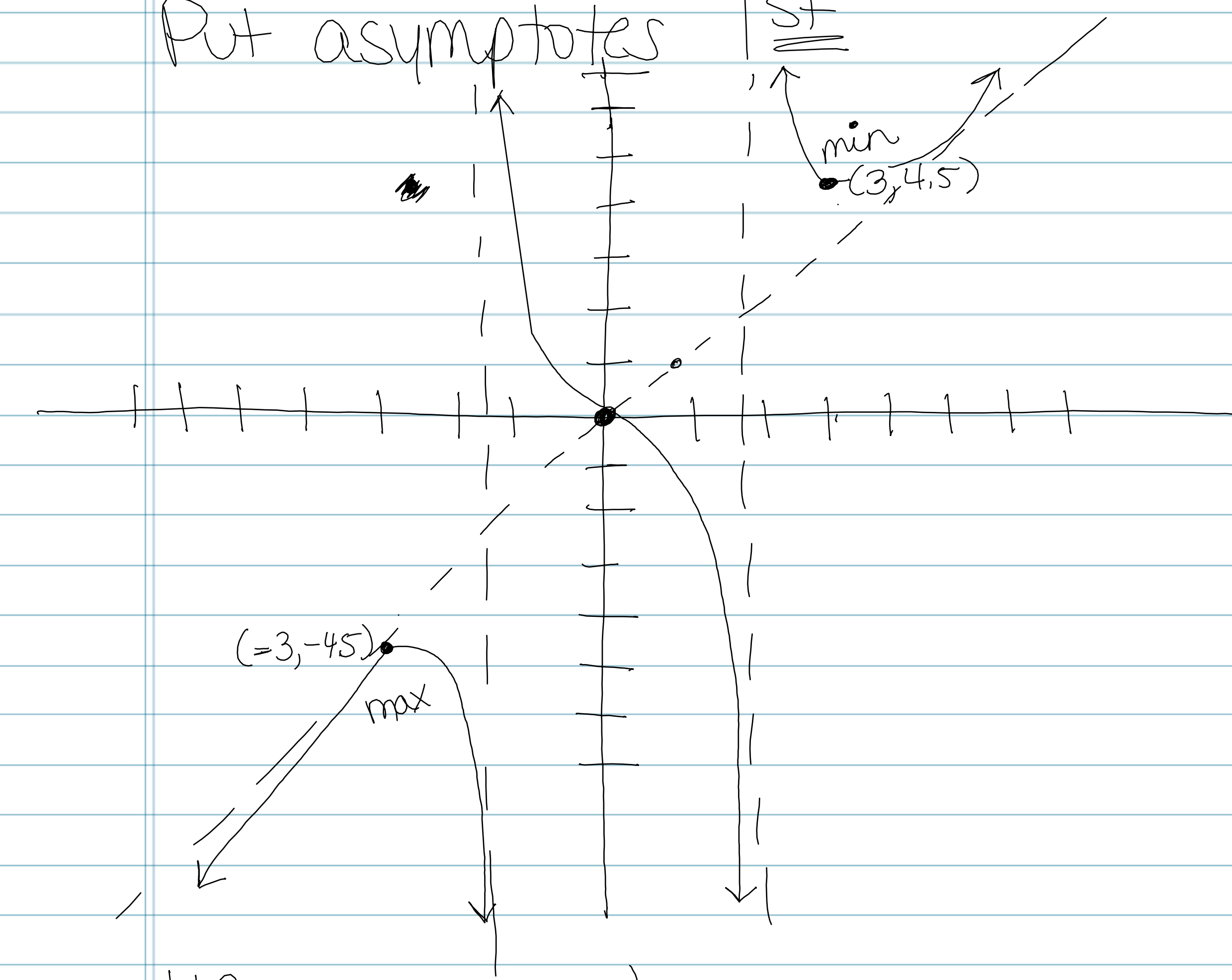
$\boxed{\text{Step 6}}$ Get points plotted on graph using $g(x)$

y,	x	y	
Stationary max	-3	-4.5	
	$-\sqrt{3}$	VA	or DNE
IP	$\varnothing$	$\varnothing$	
	$\sqrt{3}$	VA	or DNE
stationary min	3	4.5	

** If any add'l. critical points add to table, not test points **

$\boxed{\text{Graph}}$

Put asymptotes 1st



HA
$y = x$

x	y
0	0
1	1
2	2

g' + − − − − +